Tableau-based Decision Procedures for Hybrid Logic

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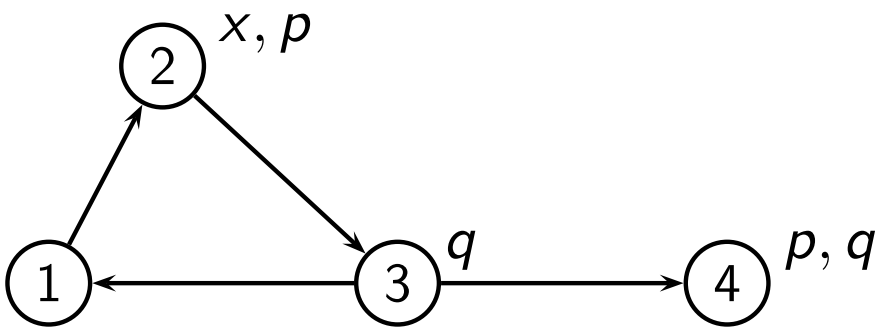
Research Goals

- Design transparent and efficient decision procedures for expressive modal languages with nominals
- Advance the art of tableaux
- Develop efficient provers

Plan of Talk

1. Models, Formulas, Tableaux
2. Prefixed Tableaux
3. Clauses and Demos
4. Clausal Tableaux
5. Final Remarks

Models



- Graphs (nodes, edges)
- Nodes are labelled with predicates (p, q, ...)
- There are predicates called nominals that can label at most one node (x, y, ...)
- NB: non-standard semantics of nominals

Modal Formulas

$$s ::= p \mid \neg s \mid s \wedge s \mid \Diamond s \mid \Diamond^* s \mid \mathsf{D} s \mid s \vee s \mid \Box s \mid \Box^* s \mid \bar{\mathsf{D}} s$$

- $M, a \models s$ in model M node a satisfies formula s
- $M, a \models \Diamond^* s$ there is a node reachable from a satisfying s
- $M, a \models \mathsf{D} s$ there is a node different from a satisfying s
- $\Diamond^*$ and $\Box^*$ are called star modalities
- D and $\bar{\mathsf{D}}$ are called difference modalities
- Formulas containing nominals are called hybrid
- We mostly assume negation normal form ($\neg p$)

- Formulas of the form $\Diamond^* s$ are called eventualities
- Eventualities cause non-compactness:
  $\Diamond^* \neg p$, p, $\Box p$, $\Box\Box p$, ...
- Difference modalities can express
  global modalities and nominals
  - Every node satisfies s: $s \wedge \bar{\mathsf{D}} s$
  - Some node satisfies s: $s \vee \mathsf{D} s$
  - At most one node satisfies s: $\bar{\mathsf{D}} \neg s \vee \mathsf{D}\bar{\mathsf{D}} \neg s$

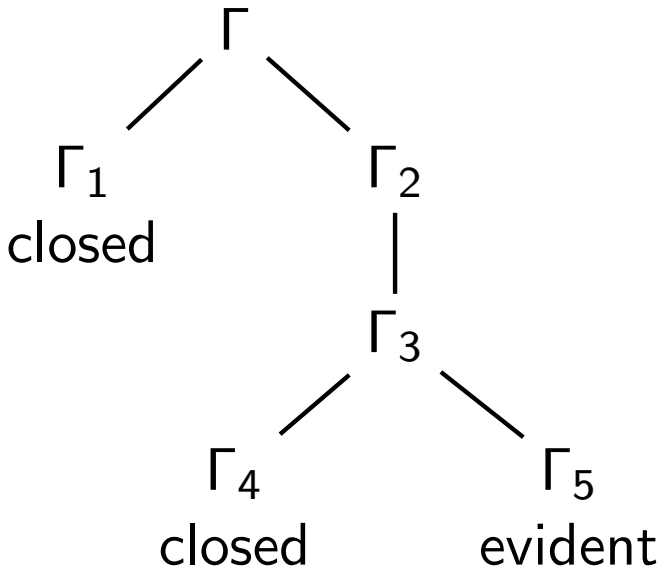
Complexity of Satisfiabilty

- Formula s is satisfiable if $M, a \models s$ for some M and a
- K is PSPACE-complete
- H is PSPACE-complete
- K with $\Box^*$ is EXP-complete ($\approx$ ALC)
- H with $\Box^*$ and $\Diamond^*$ is EXP-complete (hybrid μ-calculus)

Wanted: Constructive Decision Procedures

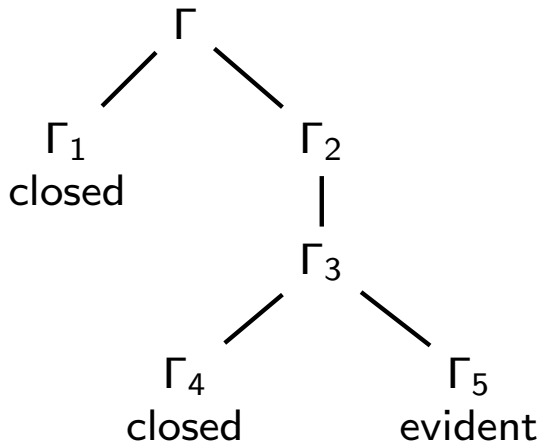
- Given a formula s,
  - return a finite model of s if s is satisfiable
  - return "unsatisfiable" if s is unsatisfiable
- Procedures should
  - elegant (e.g., transparent correctness proof)
  - be practical (goal-directed, incremental), see reasoners for description logics

Method: Tableau Systems



- A branch is a finite, nonempty set Γ of formulas
- $M \models \Gamma$ iff $\forall\, s \in \Gamma\ \exists\, a.\ M, a \models s$
- Expansion rules add formulas to branch such that satisfiability is preserved
- Closing rules identify unsatisfiable branches
- A branch is evident if no rules applies to it

Correctness of Tableau Systems



- Termination Tableau construction terminates
- Soundness Satisfiable branches are either evident or have a satisfiable expansion
- Completeness Evident branches are finitely satisfiable
- Correct tableau system describes a tableau construction procedure that yields a constructive decision procedure
- Nondeterminism There may be many complete tableaux for a given initial branch; may differ in size; each of them decides satisfiability of initial branch

Design Space for Tableau Systems

- Which formulas?
- Which notion of evidence?
- Which rules?

II Prefixed Tableaux

- Originated with Kripke 1963
- Previous work on prefixed tableaux for hybrid logic
  - Bolander and Braüner, J. Log. Comput. 2006
  - Bolander and Blackburn, J. Log. Comput. 2007
  - Horrocks and Sattler, JAR 2007
- Our work (Kaminski and Smolka) considers hybrid logic with difference modalities, graded modalities, star modalities and transitive relations
  - HyLo 2007, M4M 2007, IJCAR 2008, JoLLI 2009, Tableaux 2009, TCS 2010
  - Spartacus prover for H with global modalities: M4M 2009, ENTCS 2010
- Here: H with $\Box^*$, D, $\bar{\mathrm{D}}$

Prefixed Formulas

$$x : s$$

- x is a prefix, s is a modal formula
- Prefixes name the nodes of the model to be constructed
- We represent prefixes as nominals
- $M \models x : s$ iff M has a node labeled with x that satisfies s
- Invariant for tableau expansion: All modal formulas are subformulas of the initial modal formulas
- Prefixed tableau system terminates if number of prefixes can be bounded

Four Kinds Prefixed Formulas

$$
\begin{array}{rcl}
x : s & \rightsquigarrow & x \wedge s \\
rxy & \rightsquigarrow & x \wedge \Diamond y \\
x = y & \rightsquigarrow & x \wedge y \\
x \neq y & \rightsquigarrow & x \wedge \neg y,\ y \wedge \neg x
\end{array}
$$

- Branch is a set of prefixed formulas
- A model satisfies a branch
  if it satisfies every formula of the branch
- A model satisfies a modal formula
  if it has a node that satisfies the formula
- Hybrid logic can internalize prefixed formulas
- Prefixes simplify formulation and analysis of tableau system

Tableau Rules for K with $\Box^*$

$$\frac{x : s \ , \ x : \neg s}{\text{closed}} \qquad \frac{x : s \wedge t}{x : s \ , \ x : t} \qquad \frac{x : s \vee t}{x : s \mid x : t}$$

$$\frac{x : \Box s \ , \ rxy}{y : s} \qquad \frac{x : \Box^* s}{x : s \ , \ x : \Box\Box^* s} \qquad \frac{x : \Diamond s}{rxy \ , \ y : s} \ y \text{ fresh}$$

- Diamond rule is blocked if evidence condition for $x : s$ is satisfied

$$\frac{x : \Diamond s \ , \ x : \Box s_1 \ , \ldots , \ x : \Box s_n}{ryz \ , \ z : s \ , \ y : \Box s_1 \ , \ldots , \ y : \Box s_n}$$

- Ensures termination since there are only finitely many patterns $\Diamond s, \Box s_1, \ldots, \Box s_n$
- Pattern-based blocking [HyLo 2007], implemented in Spartacus

Model Construction

- Construct model for evident branch

$$\frac{x : \Diamond s \;\;,\;\; x : \Box s_1 \;\;, \ldots ,\;\; x : \Box s_n}{ryz \;\;,\;\; z : s \;\;,\;\; y : \Box s_1 \;\;, \ldots ,\;\; y : \Box s_n} \qquad \frac{x : \Box s \;\;,\;\; rxy}{y : s}$$

- Nodes = prefixes of evident branch
- Edges = pairs (x, y) such that $\forall s.\ x : \Box s \;\Rightarrow\; y : s$
  (i.e., all edges that respect box formulas of branch)

Extension to Nominals

- A prefixed formula $x : y$ is an equational constraint $x = y$
- Work with nominal equivalence, that is, least equivalence relation $\sim$ such that $x \sim y$ if $x : y$ or $x = y$ on the branch
- Lift tableau rules to equivalence classes

$$\frac{\tilde{x} : s \ , \ \tilde{x} : \neg s}{\text{closed}} \qquad \frac{\tilde{x} : s \wedge t}{x : s \ , \ x : t} \qquad \ldots$$

- One additional rule $\dfrac{\tilde{x} : \neg x}{\text{closed}}$
- Model construction
  - Nodes = equivalence classes of prefixes
  - Edges = $(\tilde{x}, \tilde{y})$ such that $\forall s.\ \tilde{x} : \Box s \ \Rightarrow \ \tilde{y} : s$
- Straigthforward implementation, see Spartacus

Rules for Difference Modalities

$$\frac{x : \mathsf{D}s}{y : s \;,\; y \neq x}\; y \text{ fresh} \qquad \frac{x : \mathsf{D}s}{\;y : s \;,\; y \approx x\;}\; \qquad \frac{x \neq y}{\text{closed}}\; x \sim y$$

$$\frac{x : \bar{\mathsf{D}}s}{y = x \;\mid\; y : s}\; \text{forall prefixes } y \text{ on branch}$$

- Nominal equivalence $\sim$ essential for evidence condition for D
- Disequations $y \neq x$ are essential for termination
- At most two fresh prefixes per formula Ds
- Equations $y = x$ are essential for soundness

III Clauses and Demos

- Foundation for prefix-free decision procedures [IJCAR 2010]
- Here we consider H* (H with $\Box^*$ and $\Diamond^*$).
- Extends to hybrid PDL and difference modalities

DNF

$$s \equiv \bigvee \left(\bigwedge \text{literal} \right)$$

$$\text{literal} := p \mid \neg p \mid \Diamond s \mid \Box s$$

$$\Diamond^* s \equiv s \vee \Diamond\Diamond^* s$$

$$\Box^* s \equiv s \wedge \Box\Box^* s$$

- Clause : set of literals, no complementary pair p, $\neg p$
- Every formula can be represented as a set of clauses
- NB: Clauses are interpreted conjunctively

DNF Procedure

- We assume a DNF procedure $\mathcal{D}$ that, given a set of formulas A, yields a set of clauses $\mathcal{D}A$ such that

$$\bigwedge_{s \in A} s \quad \equiv \quad \bigvee_{C \in \mathcal{D}A} \bigwedge_{s \in C} s$$

- DNF procedure provides local propositional reasoning

Request of a Clause

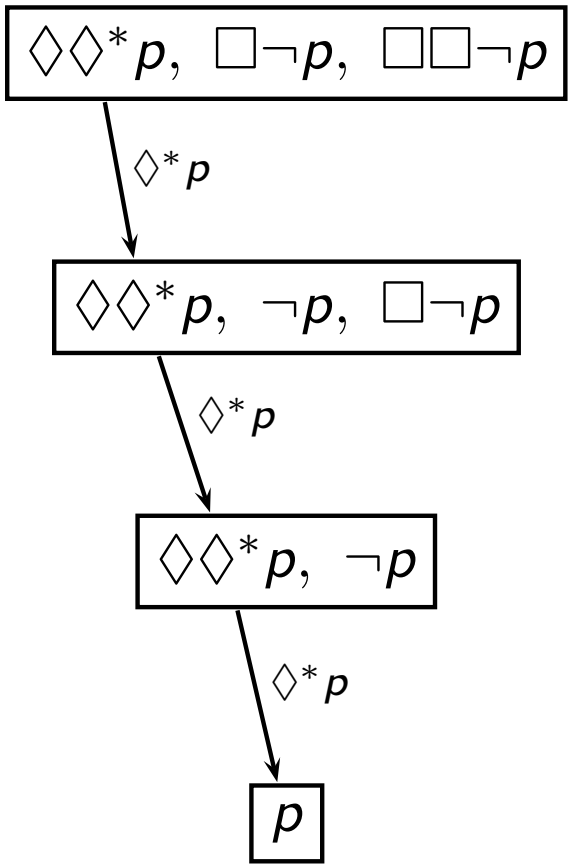
$$\mathcal{R}C \;:=\; \{\, s \mid \Box s \in C \,\}$$

- If a node satisfies C,
  then every successor of the node must satisfy $\mathcal{R}C$
- If a node satisfies C and $\Diamond s \in C$,
  then the node must have a successor that satisfies
  a clause $D \in \mathcal{D}(\mathcal{R}C; s)$

Demos

- Demos are syntactic models
- Nodes of demos are clauses such that $\Delta, C \models C$
- Edges of demos are described as links CsD that identify the literal $\Diamond s \in C$ they satisfy

Example: Construction of a Demo



Note: $\Diamond^* p \equiv p \vee \Diamond\Diamond^* p$

Links

- Minimal link: Triple CsD such that $\Diamond s \in C$ and $D \in \mathcal{D}(\mathcal{R}C; s)$
- Lifted Link: Triple CsD such that CsD' is minimal link for some $D' \subsetneq D$
- Lifted links are needed to accommodate nominals

Definition of Demos

- A demo is a finite, nonempty set of clauses and links such that

$$\frac{\Diamond s \in C}{CsD} \qquad \frac{CsD}{C\,,D} \qquad \frac{x \in C\,, x \in D}{C = D}$$

$$\frac{\Diamond\Diamond^* s \in C}{\Diamond^* s\text{-path from } C \text{ to } D \text{ such that } D \rhd s}$$

- $D \rhd s \;:\Leftrightarrow\; \exists C \in \mathcal{D}\{s\}.\ C \subseteq D$ $\qquad$ D supports s
- A demo is a model (nodes = clauses, edges = links)
- A demo Δ satisfies Δ, $C \models C$ for all nodes / clauses

Finite Supply of Literals

- When we construct a demo for a formula s, it suffices to consider a finite set $\mathcal{L}s$ of literals that can be computed in linear time; this leaves us with a finite search space
- A literal base is finite set $\mathcal{L}$ of literals closed under taking minimal links:

$$\forall\, C \subseteq \mathcal{L} \;\; \forall\, \Diamond s \in C \;\; \forall\, D \in \mathcal{D}(\mathcal{R}C; s).\;\; D \subseteq \mathcal{L}$$

- For every formula s one can obtain in linear time a literal base $\mathcal{L}s$ containing the clauses of $\mathcal{D}\{s\}$
- $\mathcal{L}s$ basically consists of the literals occurring as subformulas in s

Demo Theorem

For every satisfiable formula s
there exists a demo satisfying s
that employs only literals from $\mathcal{L}s$.

- Small model theorem
- Yields naive decision procedure
- Proof for K*
  - Let M be model of s
  - All clauses $C \subseteq \mathcal{L}s$ satisfied by M
  - All links between these clauses

IV Clausal Tableaux

- Take clauses and links as formulas
- Construct demos
- Here: Clausal decision procedure for H* [IJCAR 2010]
- Extends to hybrid PDL

- The term "clausal tableaux" has been used before for a rather different approach by Nguyen and Goré [1999, 2009]

Clausal Tableaux for K*

- A branch is a finite, nonempty set of clauses and links such that:

$$\frac{CsD}{C\,,D} \qquad \frac{CsD\,,CsD'}{D = D'}$$

- Tableaux rules

$$\frac{\Diamond s \in C}{CsD\,,D \;\mid\; \cdots}\; D \in \mathcal{D}(\mathcal{R}C; s) \qquad \frac{\Diamond s \in C}{\text{closed}}\; \mathcal{D}(\mathcal{R}C; s) = \emptyset$$

Bad loop rule $\dfrac{C_1 \overset{\Diamond^* s}{\to} \cdots \overset{\Diamond^* s}{\to} C_n \overset{\Diamond^* s}{\to} C_1}{\text{closed}}\; \forall i \in [1, n].\; C_i \not\rhd s$

where $C \overset{s}{\to} D$ means that CsD is on branch

Correctness (K*)

- Termination straightforward since all clauses are subsets of initial literal base
- Completeness straightforward since evident branches are demos (bad loop rule guarantees satisfaction of eventualities)
- Soundness challenging since one needs a semantics for star links that justifies bad loop rule
- Example
  - $C = \{\Diamond\Diamond^* p\}$ is satisfiable clause
  - $\{C, C(\Diamond^* p)C\}$ is closed branch
  - Link $C(\Diamond^* p)C$ must be unsatisfiable

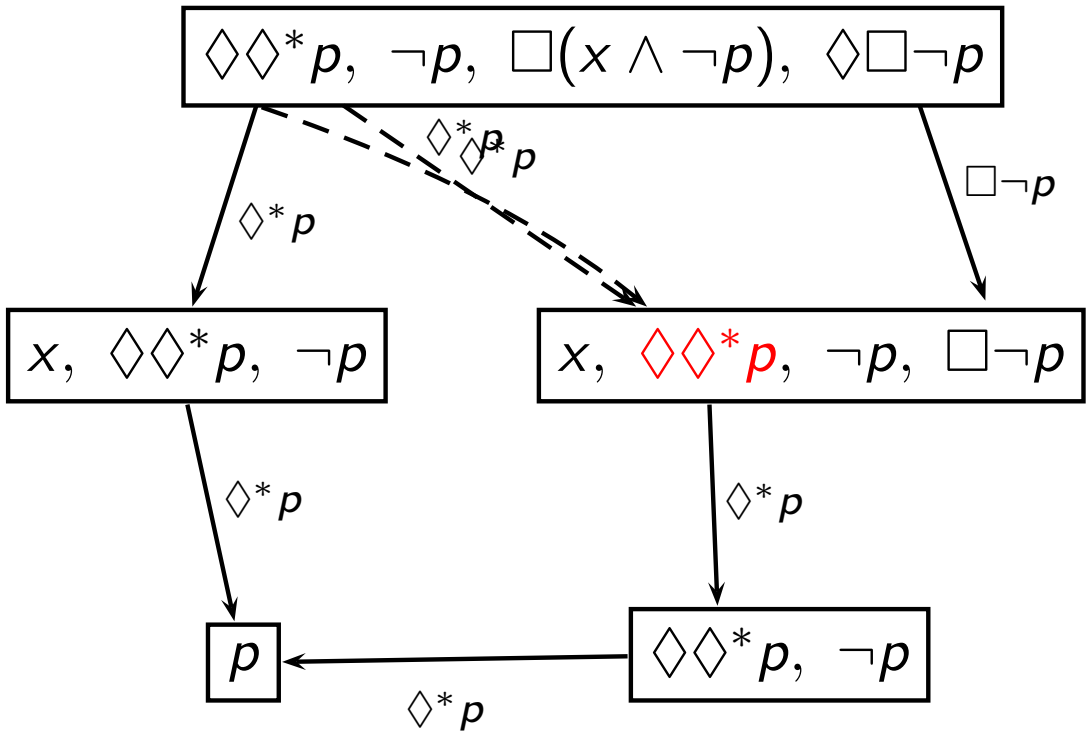
Minimal Distance Semantics for Star Links

$\delta_M As$:= minimal distance from a node satisfying A to a node satisfying s

- M satisfies $C(\Diamond^* s)D$ if
  - $\delta_M Cs > 0 \Rightarrow \delta_M Cs > \delta_M Ds$
  - $\delta_M Ds = 0 \Rightarrow D \triangleright s$

- Link must reduce minimal distance to s
- Link must deliver (i.e., $D \triangleright s$) if minimal distance is 0
- Minimal distance idea appears in [Baader 1990]

Clausal Tree Tableaux for H*, Example



Demo consists of nominally maximal clauses

Clausal Tree Tableaux for H*

- Nominal completion

$$C^\Gamma := C \cup \{\, s \mid \exists x \in C\ \exists D \in \Gamma.\ x \in D \wedge s \in D \,\}$$

- Require branches to be nominally coherent

$$\frac{C}{\overline{C^\Gamma}}$$

- Ignore clauses that aren't nominally maximal (i.e, $C = C^\Gamma$)
- See link CsD as link CsD^Γ (link lifting)

$$C \xrightarrow{s} D \ :\Leftrightarrow\ \exists E.\ CsE \in \Gamma \wedge E^\Gamma = D$$

Tableau Rules for H*

$$\frac{\Diamond s \in C}{C s D^{\Gamma}, D^{\Gamma} \mid \cdots} \quad C = C^{\Gamma},\ D \in \mathcal{D}(\mathcal{R}C; s),\ D^{\Gamma} \text{ clause}$$

$$\frac{\Diamond s \in C}{\text{closed}} \quad \forall D \in \mathcal{D}(\mathcal{R}C; s).\ D^{\Gamma} \text{ not a clause}$$

$$\frac{C_1 \overset{\Diamond^* s}{\rightarrow} \cdots \overset{\Diamond^* s}{\rightarrow} C_n \overset{\Diamond^* s}{\rightarrow} C_1}{\text{closed}} \quad \forall i \in [1, n].\ C_i \not\Diamond s$$

Correctness (H*)

- Termination: As for K*
- Soundness: As for K*, we have $\delta_M Cs = \delta_M C^\Gamma s$
- Completeness: Take clauses C with $C = C^\Gamma$

V Final Remarks

Complexity

n : size of initial formula

n : number of literals to be considered

2^n : number of clauses to be considered

2^{2^n} : number of branches to be considered

- H* satisfiability is in Exp
- Must not construct complete tableaux in tree representation
- Must avoid recomputation at clause level
- Switch to graph representation to stay in EXP
  - [Pratt 1980] PDL
  - [Goré and Widmann, IJCAR 2010] PDL with converse

Graph Representation and Nominals

- Graph representation is straightforward for K* if eventuality checking is done at end
- Yields EXPTIME decision procedure
- Nominals cause severe complications, no good solution so far
- Satisfiability of clause must be determined under nominal assumptions and may depend on nominal assumptions.

Main Contributions

- Pattern-based blocking for prefixed tableaux
- Terminating prefixed tableaux for difference modalities
- Clauses and demos
- Decision procedure for H*